Int. J. Heat Mass Transfer. Vol. 28, No. 12, pp. 2261–2268, 1985
Printed in Great Britain

0017–9310/85 $3.00 + 0.00

Solidification in a finite, initially overheated slab

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(*Received* 18 *February* 1985 *and in final form* 17 *June* 1985)

Abstract—An approximate theory of solidification in a finite, initially overheated slab is developed for small Stefan numbers. One wall of the slab is taken to be insulated and the other is subject to an instantaneous temperature drop below the freezing point. Our approach combines the heat-balance integral method and the time-dependent perturbation theory. The resulting solution is valid uniformly in time. It predicts quantitatively the deviations of the process considered from solidification with no overheating. Simple expressions for the solidification time are derived. The accuracy of the present model is examined by comparing it with various asymptotic solutions.

1. INTRODUCTION

PHASE-CHANGE heat transfer arises in many problems of applied science. These problems are highly nonlinear due to the motion of the change-of-phase front. Exact solutions are known only for some few cases. The most notable among them is that of Neumann (given in ref. [1]) describing a semi-infinite one-dimensional slab, initially at a uniform temperature. Solidification (melting) of this system occurs due to a sudden temperature drop (increase) at its wall, which is kept at this temperature later on. Systems with more complicated geometry, boundary and initial conditions are treated by numerical and approximate analytic methods [2–19].

One such problem is the one-dimensional solidification in a slab of a finite thickness, one end of which is insulated, whereas the other one is subject to an instantaneous temperature drop below the freezing point. When the slab is initially at the freezing point, its solidification is described exactly by the one-phase Neumann solution. However, no exact solution is known when initial overheating exists. Solomon [18, 19] suggested that this problem could be described by the two-phase Neumann solution as long as the temperature of the insulated end remains close to its initial value. The duration of this initial stage is an order of magnitude shorter than the characteristic heat diffusion time in the liquid.

Cho and Sunderland [13] studied this problem using the heat balance integral method [10–12]. The spatial dependence of the solid temperature was assumed to be the same as for the semi-infinite slab. This assumption leads to the exact result for the one-phase case. However, it restricts the interface position to be proportional to the square-root of time. For the two-phase case this restriction is incompatible with the heat balance at the interface during the final stage of the process, which begins when the thermal front in the liquid reaches the insulated wall. Relaxing the interface heat balance condition, the authors obtained the solidification rate, which is very close to that predicted by the two-phase Neumann solution.

For the case of small Stefan numbers St_l, St_s, which characterize the ratio of the sensible to latent heat in the solid and liquid phases, this problem was reconsidered by Weinbaum and Jiji [8]. They found asymptotic perturbative solutions up to orders St_l and $St_s^{1/2}$ for the initial and final stages, respectively. The lowest-order term of both asymptotic expansions was shown to be valid uniformly. Contrary to the results of Cho and Sunderland, which are close to the two-phase Neumann solution, the Weinbaum and Jiji model predicts the final stage to be similar to the one-phase case. This leads to a shorter freezing time. When $St_l \ll St_s$ the solution of Cho and Sunderland, which is exact at the limit $St_l = 0$, is expected to be quite accurate. The Weinbaum–Jiji solution, which involves only lowest-order terms in $St_s^{1/2}$, might lead to a significant error in this case. Determination of the higher-order terms, which would increase the accuracy, becomes practically impossible due to the extreme mathematical complexity of the equations used by Weinbaum and Jiji. In the case $St_l \sim St_s$, i.e. larger overheating, the finite thickness of the slab causes a significant deviation from the two-phase Neumann solution. This case would be described by the Weinbaum–Jiji solution more adequately than by that of Cho and Sunderland.

In this paper an approximate theory of solidification in a finite, initially overheated slab is developed for small Stefan numbers. This theory treats both cases $St_l \ll St_s$ and $St_l \sim St_s$ in a unified way and describes continuously the transition from the initial diffusion dominated stage to the large-time one-phase regime. Our model combines the heat balance integral method with a time-dependent perturbation theory. Compared with the approach of Cho and Sunderland, the present analysis avoids the assumption of a square-root-type time-dependence of the interface location. The heat balance at the moving boundary is satisfied now during the entire process. The governing equations of the

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NOMENCLATURE

A	function, defined by equation (13)
c	specific heat
d	length of the slab
$E_2(z)$	exponential integral of index 2
F	function defined by equation (36)
F'	function defined by equation (42)
f	function defined by equation (17)
I	function defined by equation (33)
k	thermal conductivity
L	latent heat of fusion
St_l	Stefan number of the liquid, $(T_i - T_f)c_l/L$
St_s	Stefan number of the solid, $(T_f - T_w)c_s/L$
T	temperature
T_f	fusion temperature
T_i	initial temperature
T_w	the front wall temperature
t	time
x	distance
u	function defined by equation (30).
δ_1	position of the change-of-phase front
Δ_1	dimensionless position of the change-of-phase front, δ_1/d
Δ_2	dimensionless position of the thermal front in liquid
θ_l	dimensionless liquid temperature, $(T_l - T_f)c_l/L$
θ_s	dimensionless solid temperature, $(T_s - T_f)c_s/L$
λ_1	constant defined by equation (20)
λ_2	constant defined by equation (21)
ν	ratio of the thermal diffusivities of solid and liquid, α_s/α_l
ρ	density
τ	dimensionless time, $t\alpha_l/d^2$
$\bar{\tau}$	dimensionless time by which the thermal front reaches the insulated wall
τ^*	dimensionless solidification time
ξ	dimensionless distance, x/d.

Greek symbols

α	thermal diffusivity
$\Gamma(a, z)$	incomplete gamma function

Subscripts

l	liquid
s	solid.

present model are less accurate than the exact partial differential equations used by Weinbaum and Jiji. Nonetheless, it becomes possible to go beyond the lowest-order asymptotic expansions and to construct the uniformly valid solution satisfying the heat balance equations to higher orders than in the model of Weinbaum and Jiji. In the case $St_l \sim St_s$ our results agree with those of Weinbaum and Jiji up to the order St_l, whereas in the case $St_l \ll St_s$ our accuracy is higher. These results also validate the applicability of the Cho–Sunderland method for solidification with small initial overheating.

2. FORMULATION OF THE PROBLEM

The system considered is a uniform liquid layer initially at the temperature T_i which is above the freezing point T_f. The thickness of the slab is d and its back wall is insulated. At time $t = 0$ the temperature of the front wall drops to the value $T_w (T_w < T_f)$ and solidification begins. It is assumed that T_w is constant during the entire process. It is also assumed that the thermophysical parameters of each phase are constant and the densities of liquid and solid are equal. The convection effects are neglected, and the heat transfer is assumed to be one-dimensional.

The heat conduction equations for each phase are given by

$$(\partial T_s/\partial t) = \alpha_s(\partial^2 T_s/\partial x^2),$$
$$(\partial T_l/\partial t) = \alpha_l(\partial^2 T_l/\partial x^2). \quad (1)$$

The energy balance at the liquid–solid interface is defined as

$$\rho L(\mathrm{d}\delta_1/\mathrm{d}t) = k_s(\partial T_s/\partial x)|_{x=\delta_1} - k_l(\partial T_l/\partial x)|_{x=\delta_1}. \quad (2)$$

The other boundary and initial conditions can be stated as

$$T_s(0, t) = T_w, \quad T_s(\delta_1, t) = T_l(\delta_1, t) = T_f,$$
$$(\partial T_l/\partial x)|_{x=d} = 0, \quad (3)$$
$$T_l(x, 0) = T_i, \quad \delta_1(0) = 0. \quad (4)$$

Introducing the dimensionless variables and parameters listed in Nomenclature these equations can be written as

$$(\partial\theta_s/\partial\tau) = \nu(\partial^2\theta_s/\partial\xi^2), \quad (\partial\theta_l/\partial\tau) = (\partial^2\theta_l/\partial\xi^2), \quad (5)$$
$$(\mathrm{d}\Delta_1/\mathrm{d}\tau) = -(\partial\theta_l/\partial\xi)|_{\xi=\Delta_1} + \nu(\partial\theta_s/\partial\xi)|_{\xi=\Delta_1}, \quad (6)$$
$$\theta_s(0, \tau) = -St_s, \quad \theta_s(\Delta_1, \tau) = \theta_l(\Delta_1, \tau) = 0,$$
$$(\partial\theta_l/\partial\xi)|_{\xi=1} = 0, \quad (7)$$
$$\theta_l(\xi, 0) = St_l, \quad \Delta_1(0) = 0. \quad (8)$$

Since equations (5)–(8) cannot be solved exactly except for the case $St_l = 0$, we develop the heat balance formulation of this problem. Following Cho and Sunderland we introduce a thermal penetration thickness Δ_2, assuming that no heat flow takes place at $\xi \geqslant \Delta_2$:

$$\theta_l|_{\xi \geqslant \Delta_2} = St_l, \quad (\partial\theta_l/\partial\xi)|_{\xi \geqslant \Delta_2} = 0. \quad (9)$$

It is also assumed that $\Delta_2(0) = 0$, so that the change-of-phase front Δ_1 and the thermal front Δ_2 begin to propagate simultaneously towards the insulated wall. Integrating the heat conduction equations (5) one obtains:

$$\mathrm{d}\left(\int_0^{\Delta_1} \theta_s \,\mathrm{d}\xi\right)\Big/\mathrm{d}\tau = \nu[(\partial\theta_s/\partial\xi)|_{\xi=\Delta_1} - (\partial\theta_s/\partial\xi)|_{\xi=0}], \tag{10}$$

$$\mathrm{d}\left(\int_{\Delta_1}^{\Delta_2} \theta_l \,\mathrm{d}\xi\right)\Big/\mathrm{d}\tau = St_l(\mathrm{d}\Delta_2/\mathrm{d}\tau) - (\partial\theta_l/\partial\xi)|_{\xi=\Delta_1}. \tag{11}$$

We now assume that the spatial temperature distribution in the liquid could be approximated by the parabolic profile:

$$\theta_l = \begin{cases} St_l[1-(\Delta_2-\xi)^2(\Delta_2-\Delta_1)^{-2}], & \Delta_1 \leqslant \xi \leqslant \Delta_2 \\ St_l, & \Delta_2 \leqslant \xi \leqslant 1. \end{cases} \tag{12}$$

It satisfies the boundary conditions (7) and (9). Similarly the temperature in the solid phase is chosen as

$$\theta_s = St_s[A(\Delta_1-\xi) - (1+\Delta_1 A)\Delta_1^{-2}(\Delta_1-\xi)^2], \tag{13}$$

where $A(\tau)$ is some unknown function of time. The temperature given by equation (13) satisfies the boundary conditions (7). Substituting equations (12) and (13) into (6), (10) and (11) one obtains the system of coupled ordinary differential equations for the functions $\Delta_1(\tau)$, $\Delta_2(\tau)$ and $A(\tau)$:

$$A = -\Delta_1^{-1} - (6\nu)^{-1}[2^{-1}\Delta_1^2(\mathrm{d}A/\mathrm{d}\Delta_1) + A\Delta_1 - 1] \times (\mathrm{d}\Delta_1/\mathrm{d}\tau), \tag{14}$$

$$2(\mathrm{d}\Delta_1/\mathrm{d}\tau) + (\mathrm{d}\Delta_2/\mathrm{d}\tau) = 6(\Delta_2-\Delta_1)^{-1}, \tag{15}$$

$$\mathrm{d}\Delta_1/\mathrm{d}\tau = -A\nu St_s - 2St_l(\Delta_2-\Delta_1)^{-1}. \tag{16}$$

with the initial conditions $\Delta_1(0) = \Delta_2(0) = 0$. The initial value of $A(\tau)$ is not defined due to the sudden temperature drop at the front wall at $t = 0$.

At $\tau = \bar{\tau}$ the thermal penetration depth becomes equal to the thickness of the slab, $\Delta_2(\bar{\tau}) = 1$ and the next stage of the process begins. For this stage the liquid temperature is assumed to be

$$\theta_l = St_l f(1-\bar{\Delta}_1)^{-2}[(1-\Delta_1)^2 - (1-\xi)^2], \quad \bar{\tau} \leqslant \tau \leqslant \tau^*. \tag{17}$$

Here f is some unknown function of time, defining the relaxation of the liquid temperature, and $\bar{\Delta}_1 = \Delta_1(\bar{\tau})$. From equations (17) and (12) it follows that $f(\bar{\tau}) = 1$. The temperature given by equation (17) satisfies the boundary conditions (7). The solid temperature is taken as before according to equation (13). Then equations (6), (11), (13) and (17) lead to

$$\mathrm{d}\Delta_1/\mathrm{d}\tau = -\nu St_s A - 2St_l f(1-\Delta_1)(1-\bar{\Delta}_1)^{-2}, \quad \tau \geqslant \bar{\tau}, \tag{18}$$

$$(1-\Delta_1)[f(\mathrm{d}\Delta_1/\mathrm{d}\tau) - 3^{-1}(1-\Delta_1)(\mathrm{d}f/\mathrm{d}\tau)] = f, \quad \tau \geqslant \bar{\tau}. \tag{19}$$

These equations along with (14) and initial conditions $f(\tau) = 1$, $\Delta_1(\bar{\tau}) = \bar{\Delta}_1$ determine the freezing for $\tau \geqslant \bar{\tau}$. The present heat balance integral formulation differs from that of Cho and Sunderland by the choice of the solid temperature. This way it is possible to satisfy the heat balance at the moving change-of-phase front uniformly in time.

3. SOLUTION

We now proceed with a solution of the differential equations of the heat-balance integral method formulated above. For the initial stage $\tau \leqslant \bar{\tau}$, which resembles the Neumann problem, it is natural to assume:

$$\Delta_1 = \lambda_1\sqrt{\tau}, \quad \tau \leqslant \bar{\tau}. \tag{20}$$

Eliminating the function A from equations (15) and (16) by using (14) one obtains

$$\Delta_2 = \lambda_2\sqrt{\tau}, \quad \tau \leqslant \bar{\tau}, \tag{21}$$

where the constants λ_1 and λ_2 are determined by

$$-24\nu[\lambda_1^2 - 2\nu St_s + 4St_l\lambda_1(\lambda_2-\lambda_1)^{-1}] = 4\nu St_s\lambda_1^2 + \lambda_1^4 + 4St_l\lambda_1^3(\lambda_2-\lambda_1)^{-1}, \tag{22}$$

$$(2\lambda_1+\lambda_2)(\lambda_2-\lambda_1) = 12. \tag{23}$$

Using equations (16) and (14) one finds $A(\tau)$:

$$A(\tau) = -\lambda_1(\nu St\Delta_1)^{-1}[(\lambda_1/2) + St_l(\lambda_2-\lambda_1)^{-1}] = -(1-\lambda_1^2/12\nu)(1+\lambda_1^2/24\nu)^{-1}, \quad \tau \leqslant \bar{\tau}. \tag{24}$$

At the time $\bar{\tau} = \lambda_2^{-2}$, the thermal front reaches the insulated wall; the interface position is then $\bar{\Delta}_1 = \lambda_2/\lambda_1$.

For $St_l \ll 1, St_s \ll 1, \nu \sim 1$ equations (22) and (23) can be solved by successive approximations. The first three iterations give

$$\lambda_1 \simeq (2\nu St_s)^{1/2} - 3^{-1/2}St_l + 2^{-1}(2\nu St_s)^{-1/2} \times [(St_l^2/3) - (\nu St_s^2/2) - \nu St_s St_l], \tag{25}$$

$$\lambda_2 \simeq (12)^{1/2} - 2^{-1}(2\nu St_s)^{1/2} + St_l/2(3)^{1/2}. \tag{26}$$

Consequently one obtains

$$\bar{\tau} \simeq [1 + (\nu St_s/6)^{1/2} - (\nu St_s/24) - (St_l/6)]/12, \tag{27}$$

$$\bar{\Delta}_1 \simeq (\nu St_s/6)^{1/2} + (\nu St_s/12) - (St_l/6) - (4)^{-1}(6\nu St_s)^{-1/2}[(St_l^2/3) - (\nu St_s^2/2) + (\nu^2 St_s^2/6) - 5\nu St_s St_l/3], \tag{28}$$

$$A(\Delta_1) = -\Delta_1^{-1}(1 - St_s/4), \quad 0 \leqslant \tau \leqslant \bar{\tau}. \tag{29}$$

For $\tau \geqslant \bar{\tau}$, $St_l \neq 0$ the assumption $\Delta_1 = \lambda_1\sqrt{\tau}$ is invalid because it violates the heat balance at the moving boundary, given by equation (18). In order to describe the freezing process for $\tau \geqslant \bar{\tau}$ we develop a perturbative solution of equations (14), (18) and (19) for the case of small Stefan numbers. In this case the interface motion is slow, and in the lowest approximation the solid temperature profile given by (13) is

quasi-static, i.e. $A \simeq -\Delta_1^{-1}$. Then equation (18) gives $\Delta_1 \simeq (2\nu St_s\tau)^{1/2}$. Introducing these results into (19) gives

$$f(\Delta_1) \simeq (u/\bar{u})^{3+3/\nu St_s} \exp(\bar{u}-u),$$
$$u = 3[\nu St_s(1-\Delta_1)]^{-1}, \quad \bar{u} = 3[\nu St_s(1-\bar{\Delta}_1)]^{-1}. \tag{30}$$

We now consider the solution of equation (14) up to the orders $(\nu St_s)^{3/2}$, $St_1(\nu St_s)^{1/2}$, and $St_1^2(\nu St_s)^{-1/2}$. Such a solution is found to be

$$A(\Delta_1) \simeq -\Delta_1^{-1}(1-St_s/4), \quad \bar{\tau} \leqslant \tau \leqslant \tau^*. \tag{31}$$

in agreement with the value of A at $\tau = \bar{\tau}$, given by (29). Introducing equations (30)–(31) into (18) and integrating one has

$$\tau - \bar{\tau} \simeq (2\nu St_s)^{-1}(\Delta_1^2 - \bar{\Delta}_1^2)(1+St_s/4) + I(\Delta_1), \tag{32}$$

$$I(\Delta_1) \simeq (2St_1/3\nu St_s)(1-\bar{\Delta}_1)(1-\nu St_s/3)^{-1} \times \left\{(\nu St_s/3)\,e^{\bar{u}}\bar{u}^{-3/\nu St_s}\left[\Gamma\left(\frac{3}{\nu St_s}+1,\bar{u}\right) - \Gamma\left(\frac{3}{\nu St_s}+1,u\right)\right] + \bar{\Delta}_1 - (2\nu St_s/3) - [\Delta_1 - (2\nu St_s/3)]\,e^{\bar{u}-u}(u\bar{u}^{-1})^{3/\nu St_s}\right\}. \tag{33}$$

The above expression can be substantially simplified using the smallness of the parameter νSt_s. Applying the Edgeworth asymptotic expansion of incomplete gamma function $\Gamma(a,z)$ in terms of the error function and its derivatives [20] and using the Stirling formula one obtains

$$I(\Delta_1) \simeq (2St_1/3\nu St_s)\left\{(\pi\nu St_s/6)^{1/2}[\exp(3\bar{\Delta}_1^2/2\nu St_s)] \times \left[\operatorname{erf}\left(\sqrt{\frac{3}{\nu St_s}}\frac{\Delta_1}{1-\Delta_1}\right) - \operatorname{erf}\left(\sqrt{\frac{3}{\nu St_s}}\frac{\bar{\Delta}_1}{1-\bar{\Delta}_1}\right)\right] + \bar{\Delta}_1 - \Delta_1[\exp(\bar{u}-u)](u\bar{u}^{-1})^{3/\nu St_s}\right\}. \tag{34}$$

The interface position $\Delta_1(\tau)$ is obtained using equations (32)–(34):

$$\Delta_1(\tau) \simeq (2\nu St_s\tau)^{1/2} - St_1F(\tau) - (2\nu St_s\tau)^{1/2}St_s/8, \quad \tau^* \geqslant \tau \geqslant \bar{\tau}, \tag{35}$$

$$F(\tau) \simeq \bar{\tau}(3\tau)^{-1/2} + (2/3\sqrt{\tau})\left\{2^{-1}(\pi/3)^{1/2}\,e^{-3\bar{\tau}} \times \left[\operatorname{erf}\left(\frac{\sqrt{3\tau}}{1-\sqrt{2\nu St_s\tau}}\right) - \operatorname{erf}\left(\frac{\sqrt{3\bar{\tau}}}{1-\sqrt{2\nu St_s\bar{\tau}}}\right) + \sqrt{\tau}-\sqrt{\bar{\tau}}\exp\left[(3/\nu St)\left(\frac{1}{1-\sqrt{2\nu St_s\bar{\tau}}} - \frac{1}{1-\sqrt{2\nu St_s\tau}} + \ln\frac{1-\sqrt{2\nu St_s\bar{\tau}}}{1-\sqrt{2\nu St_s\tau}}\right)\right]\right\}. \tag{36}$$

To the lowest order the function F is given by

$$F(\tau) \simeq \tau^{-1/2}\{3^{-1/2}\bar{\tau} + (2/3)[2^{-1}(\pi/3)^{1/2}\,e^{-3\tau} \times (\operatorname{erf}\sqrt{3\tau} - \operatorname{erf}\sqrt{3\bar{\tau}})] + \sqrt{\bar{\tau}} - \sqrt{\tau}\,e^{3(\tau-\bar{\tau})}\}. \tag{37}$$

Setting $\Delta_1 = 1$ in equation (32) and using (34) one obtains the freezing time τ^*:

$$\tau^* \simeq (2\nu St_s)^{-1} + (1/8\nu) + St_1(6\nu St_s)^{-1/2}\left\{6^{-1} + (2/3) \times \left[\sqrt{\pi}\,e^{1/4}\operatorname{erfc}\left(\sqrt{\frac{3}{2\nu St_s}}\frac{\Delta_1}{1-\Delta_1}\right) + 1\right]\right\}, \tag{38}$$

or approximately

$$\tau^* \simeq (2\nu St_s)^{-1} + 0.65\,St_1(\nu St_s)^{-1/2} + (1/8\nu). \tag{39}$$

4. RESULTS AND DISCUSSION

In this section the main results of our analysis are summarized and compared with the other solutions of the problem considered.

Temperature profiles

The solid temperature θ_s is determined by equations (13), (29) and (31). In the first approximation our result for θ_s is identical to that of Weinbaum and Jiji as well as to the lowest-order terms of the Neumann and Cho–Sunderland solutions. The liquid temperature given by equations (12), (17) and (30) has the same functional form as the parabolic profile of Cho and Sunderland. However, the functions Δ_1 and Δ_2 found by these authors coincide with those obtained in the present work only in the lowest order in the parameter $(\nu St_s)^{1/2}$.

For the relaxation stage $\bar{\tau} \leqslant \tau \leqslant \tau^*$ the thermal gradients in the liquid phase are determined by the temperature of the insulated wall $St_1f(\bar{u}/u)^2$. This temperature drops by an order of magnitude from its initial value when the interface reaches the point defined by the following equation:

$$(1-\Delta_1)^{-1} - (1-\bar{\Delta}_1)^{-1} + (1+\nu St_s/3) \times \ln[(1-\Delta_1)/(1-\bar{\Delta}_1)] = \nu St_s. \tag{40}$$

The lowest-order solution of this equation gives $\Delta_1 \simeq (2\nu St_s\tau)^{1/2}$. This means that the insulated wall temperature drops by an order of magnitude at the time which is equal to the characteristic heat diffusion time in the liquid. The model of Weinbaum and Jiji also leads to this result. However, the approximate solutions of equation (40), which involve higher powers of the parameters νSt_s and St_1, yield a shorter temperature relaxation time. The time-dependence of the insulated wall temperatures predicted by our model and by the Weinbaum–Jiji solution are compared in Fig. 1 for the case $\nu = 1$, $St_1 = St_s = 0.1$.

Interface motion

The instantaneous position of the interface $\Delta_1(\tau)$ is determined by equations (20), (25) and (35)–(37). Prior to the moment $\tau = \bar{\tau}$ the finite size effects are negligible.

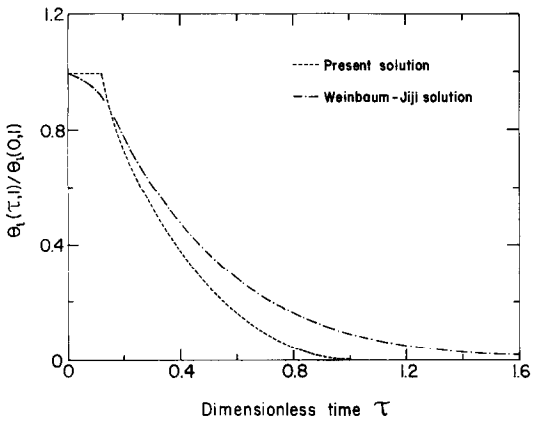

FIG. 1. Temperature–time history of the insulated wall ($v = 1$, $St_s = 0.1$, $St_l = 0.1$).

The accuracy of our results for this stage can be estimated by comparing (25) with the expansion of λ_1 corresponding to the Neumann solution. The latter expansion has the following form:

$$\lambda_1 \simeq (2vSt_s)^{1/2} - St_l\pi^{-1/2} + (2vSt_s)^{-1/2} \times [(St_l^2/2\pi) - (2vSt_sSt_l/\pi) - vSt_s^2/3]. \quad (41)$$

The leading terms in equations (25) and (41) are identical. The first corrections proportional to St_l are practically the same. Their ratio is given by $(\pi/3)^{1/2}$. The higher-order terms are also sufficiently close to each other.

For $\tau \geqslant \bar{\tau}$ the motion of the interface is given by equations (35)–(37). The first term in equation (35) corresponds to the quasi-static approximation. The next term St_lF describes the combined effects of finite thickness and initial overheating, whereas the third term gives the correction, which is independent of the initial overheating.

In the case of a very small initial overheating ($v^{1/2}St_s^{3/2} \gg St_l$), the second term in equation (35) can be ignored. Then (35) gives $\Delta_1(\tau)$ which is similar to that predicted by the one-phase Neumann solution and by the solution of Cho and Sunderland. The accuracy of our result for this case can be estimated by comparing the corrections to the leading term $(2vSt_s\tau)^{1/2}$. According to (41), the correction corresponding to the one-phase Neumann solution is $(2v\tau)^{1/2}St_s^{3/2}/6$, whereas in our model the corresponding term is $(2v\tau)^{1/2}St_s^{3/2}/8$.

When the overheating is comparable to the drop of the front wall temperature ($St_l \sim vSt_s$), the motion of the interface for $\tau \geqslant \bar{\tau}$ is significantly different from that described by the two-phase Neumann solution. As follows from equations (35)–(37), $\Delta_1(\tau)$ is no longer proportional to the square-root of time. The function $F(\tau)$ grows only till the moment when the insulated wall temperature drops by an order of magnitude from its initial value. Later on $F(\tau)$ decreases. If both vSt_s and St_l are very small, the solidification time is much larger than the temperature relaxation time. For this case the process reaches the one-phase stage, during which $St_lF(\tau) < (2v\tau)^{1/2}St_s^{3/2}/8$). When vSt_s and St_l are only moderately small, the correction corresponding to both the second and the third terms in (35) are of the same order at $\tau = \tau^*$.

The accuracy of our predictions for the function F can be examined by comparing equations (36) and (37) with the corresponding term of the Weinbaum–Jiji solution. The latter term has the form $F'St_l$, where F' can be represented by

$$F'(\tau) = -(\tau/\pi)^{1/2}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n E_2(n^2/\tau)\right], \quad \tau \leqslant \bar{\tau},\ 0 \leqslant \bar{\tau} \leqslant \tau^*, \quad (42a)$$

$$F'(\tau) = -(\bar{\tau}/\tau)^{1/2}F'(\bar{\tau}) - \sum_{n=1}^{\infty}(2n+1)^{-2} \times \{\exp[-(2n+1)^2\pi^2\bar{\tau}/4]\}\{\bar{\tau}^{1/2} - \tau^{1/2} \times \{\exp[-(2n+1)\pi^2(\tau-\bar{\tau})/4]\} + \pi^{-1/2}(2n+1)^{-1}\{\exp[(2n+1)^2\pi^2\bar{\tau}/4]\} \times [\mathrm{erf}((2n+1)\pi\sqrt{\tau}/2) - \mathrm{erf}((2n+1)\pi\sqrt{\bar{\tau}}/2)]\} \times 8\pi^{-2}\tau^{-1/2}, \quad \bar{\tau} \leqslant \tau \leqslant \tau^*. \quad (42b)$$

Here $E_2(z)$ is the exponential integral of index 2 [20]. The term St_lF' is the only correction to the leading term $(2vSt_s\tau)^{1/2}$ accounted for in the Weinbaum–Jiji solution. The function $F(\tau)$ given by (37) and the function $F'(\tau)$ are shown in Fig. 2. Their numerical values are practically the same. According to equation (36) the higher-order corrections to the function F lead to a sharper maximum, which is reached earlier than that of functions F given by (37) and (42). The effect of these corrections on the time-dependence of the F-function is illustrated in Fig. 2. In Fig. 3 the instantaneous location of the interface as a function of time is presented for the case $v = 1, St_s = St_l = 0.1$. The curve predicted by our solution is bounded from below and above by the curves corresponding to the two-phase and one-phase Neumann solutions, respectively. Our solution starts as the two-phase Neumann solution and tends to the one-phase solution in the course of the process.

Solidification time

The solidification time is defined by equations (38) and (39). They involve two corrections to the leading term $(2vSt_s)^{-1}$. The first of them is independent of the overheating level. The second one, which involves $St_l(vSt_s)^{-1/2}$, describes the effect of overheating. These corrections are quite different from those occurring in the two-phase Neumann solution. Consequently, our model predicts a shorter freezing time than that for an equivalent piece of a semi-infinite slab.

The comparison of various solutions is presented in Fig. 4, which gives the solidification time for $v = 1$, $St_s = 0.3$ and St_l varying from 0 to 0.3. For $St_l = 0.3$ the ratio of the difference between the solidification times corresponding to the two-phase and one-phase

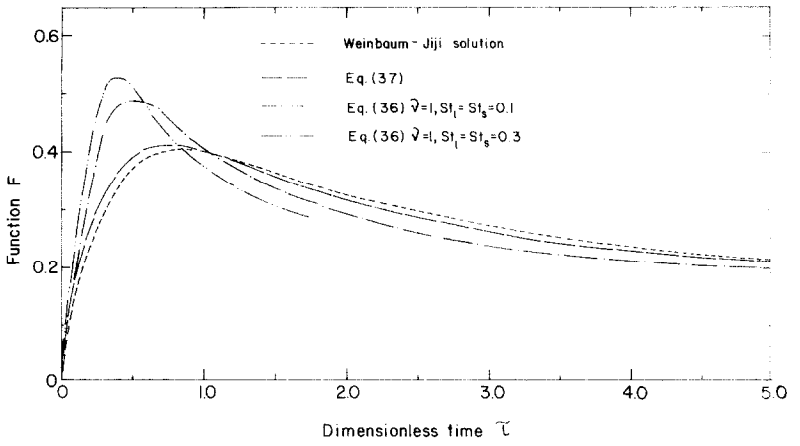

FIG. 2. Time dependence of the F-functions.

Neumann solutions to their average is about 25%. The Cho–Sunderland solution with a parabolic θ_l profile gives τ^* which is slightly larger than that predicted by the two-phase Neumann solution. Another solution of Cho and Sunderland, which assumes the trigonometric profile for θ_l gives τ^*, which is shorter than that corresponding to the two-phase Neumann solution. For small values of St_l the Weinbaum–Jiji model yields τ^*, which is shorter than that without initial overheating. This deviation is about 9% for $St_l = 0$. In the same case our solution involves an error which is about 2%. The slopes of the lines corresponding to equation (39) and the Weinbaum–Jiji solutions are practically the same. A more accurate result corresponds to (38). It predicts a slightly shorter solidification time.

5. CONCLUDING REMARKS

In this paper an approximate theory of solidification in a finite, initially overheated slab is presented for the case of small Stefan numbers. The solidification process is described as a transition from an initial two-phase stage to the regime in which the temperature gradients in the liquid are negligible. The combined effects of the initial overheating and the finite thickness lead to a deviation from the $\Delta_1 \sim \lambda_1 \tau^{1/2}$ law, which characterizes the interface motion in a semi-infinite system. This deviation is not accounted for in the work of Cho and Sunderland. According to our model the solidification rate of the system is faster than that corresponding to the semi-infinite slab with the same initial overheating, but slower than in a slab without overheating.

Our solution, which is valid uniformly in time, and which gives $\Delta_1(\tau)$ up to the orders $\nu^{1/2}St^{3/2}$, $St_l(\nu St_s)^{1/2}$ and $St_l^2(\nu St_s)^{-1/2}$ predicts Δ_1 to be proportional to the square-root of time only in the leading order $(\nu St_s)^{1/2}$. Up to the order St_l our model gives practically the same result for $\Delta_1(\tau)$ as the solution of Weinbaum and Jiji. Yet it shows that the higher-order terms are important at the final stage of the process, when the temperature of the liquid has already dropped by an order of magnitude from its initial value. For $\nu St_s \sim St_l$ the contribution of these terms to the total solidification time is of the same order as of those accounted for by

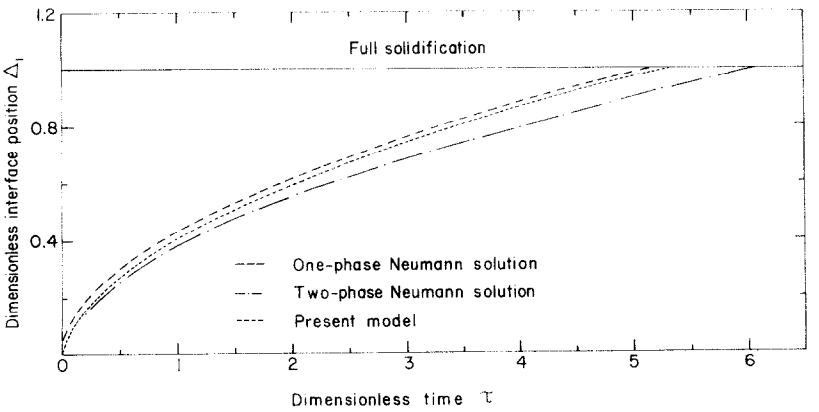

FIG. 3. Position of the change-of-phase front as a function of time ($\nu = 1$, $St_s = 0.1$, $St_l = 0.1$).

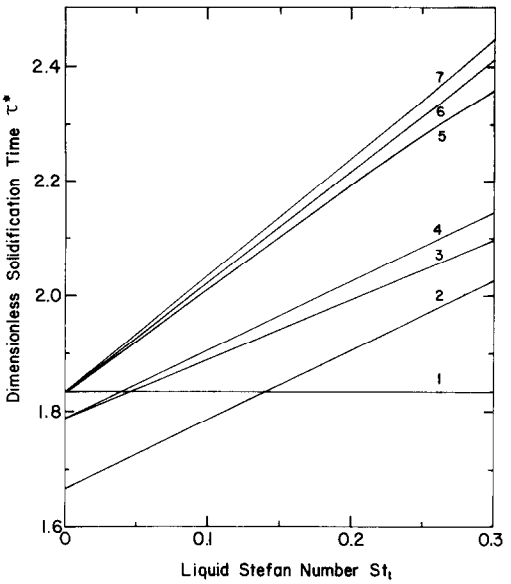

FIG. 4. Solidification time as a function of the liquid Stefan number ($v = 1$, $St_s = 0.3$). The curves labeled by numbers 1–7 correspond to the following solutions: 1, one-phase Neumann solution; 2, Weinbaum–Jiji solution; 3, present model equation (38); 4, present model equation (39); 5, Cho–Sunderland solution with the trigonometric profile; 6, two-phase Neumann solution; 7, Cho–Sunderland solution with the parabolic profile.

Weinbaum and Jiji. When $vSt_s \gg St_l$ the term involving $v^{1/2}St_s^{3/2}$ in $\Delta_1(\tau)$ represents the main correction to the leading term and cannot be ignored. This is the case when the two-phase and one-phase Neumann solutions, as well as the Cho–Sunderland model converge. The numerical coefficient of the correction term proportional to $v^{1/2}St_s^{3/2}$ in our model is not accurate enough leading to an error of the order 2% in the solidification time for $St_l = 0$.

One of the main results of the present work is the simple expression derived for the solidification time as given by equation (39). This expression can be useful for practical estimates of the process in finite systems which were studied so far using the two-phase and one-phase Neumann solutions [18, 19].

Acknowledgements—The authors are grateful to Professors Y. Zarmi, L. Gutman, D. Faiman, A. Nir, D. Govaer, Dr J. Gordon and especially Dr A. Zemel, for important discussions and suggestions. We are also grateful to Dr Z. Ofer and Mrs L. Naaman for technical assistance.

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SOLIDIFICATION D'UNE COUCHE FINIE ET INITIALEMENT SURCHAUFFEE

Résumé—Une théorie approchée de la solidification d'une plaque finie, initialement surchauffée est développée pour des petits nombres de Stefan. Une paroi de la plaque est isolée et l'autre est soumise à une baisse brutale de température au dessous du point de solidification. Cette approche combine la méthode du bilan intégral et la théorie de perturbation variable dans le temps. Elle prédit quantitativement les déviations du process considéré par rapport à la solidification sans surchauffe. Des expressions simples pour le temps de solidification sont obtenues. La précision du modèle est examinée par comparaison avec différentes solutions asympotiques.

VERFESTIGUNG IN EINEM ENDLICHEN, ANFANGS ÜBERHITZTEN SPALT

Zusammenfassung—Für die Verfestigung in einem endlichen, anfangs überhitzten Spalt wird für kleine Stefanzahlen eine Theorie entwickelt. Eine Wand des Spaltes wird als isoliert angenommen, die andere wird einer sprunghaften Temperaturabsenkung bis unter den Gefrierpunkt unterworfen. Unser Verfahren kombiniert die Wärmebilanz-Integral-Methode und die zeitabhängige Störtheorie. Die daraus resultierende Lösung ist über die gesamte Zeit gültig. Es berechnet quantitativ die Abweichung des hier betrachteten Prozesses von der Verfestigung ohne Überhitzung. Einfache Ausdrücke für die Verfestigungszeit werden hergeleitet. Die Genauigkeit des vorgestellten Modells wird durch Vergleiche mit verschiedenen asymptotischen Lösungen untersucht.

ЗАТВЕРДЕВАНИЕ ПЕРВОНАЧАЛЬНО ПЕРЕГРЕТОЙ ОГРАНИЧЕННОЙ ПЛАСТИНЫ

Аннотация—Для малых чисел Стефана развита приближенная теория затвердевания ограниченной пластины, находящейся первоначально в перегретом состоянии. Одна поверхность пластины теплоизолирована, другая мгновенно охлаждается до температуры ниже точки кристаллизации. Принятый подход сочетает интегральный метод, основанный на балансе энергии и теорию возмущений, учитывающую зависимость от времени. Полученное решение корректно для любого момента времени. Теория показывает, что исследуемый процесс количественно отличается от процесса затвердевания, рассматриваемого без учета перегрева вещества. Получены простые соотношения для времени затвердевания пластины. Точность данной модели исследуется путем ее сравнения с различными асимптотическими решениями.